

FINAL REPORT

NASA Research Grant NAG3-1384

FINAL
11-34-CR
14 CIT
018916

Period of Performance

11/18/93-11/18/96

(includes 12 months no-cost extension 11/18/95-11/18/96)

STABILITY LIMITS AND DYNAMICS OF NONAXISYMMETRIC LIQUID BRIDGES

PREPARED BY

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NAG3-1384: Stability Limits and Dynamics of Nonaxisymmetric Liquid Bridges

Final Report

1.1 General

The research accomplishments summarized in this Final Report refers to work funded under the grant NAG3-1384 *Stability Limits and Dynamics of Nonaxisymmetric Liquid Bridges* during the period from 11/18/93 to 11/18/96, which included the 12 months no-cost extension granted at the end of the nominal 3 year period of performance. The report has 5 sections, in section 1 the objectives are presented, a task description is given and the background and significance of the work is outlined. In section 2 the research accomplishments are summarized. In section 3 publications and presentations are listed. Student participation is listed in 4 and references for sections 1 and 2 are supplied in section 5.

Objectives:

- Theoretical and experimental investigation of the stability of nonaxisymmetric and axisymmetric bridges contained between equal and unequal radii disks as a function of Bond and Weber number with emphasis on the transition from unstable axisymmetric to stable nonaxisymmetric shapes.
- Numerical analysis of the stability of nonaxisymmetric bridges between unequal disks for various orientations of the gravity vector
- Experimental and numerical investigation of bridge stability (nonaxisymmetric and axisymmetric), large amplitude (nonaxisymmetric) oscillations and breaking.

1.2 Task Description:

This project involves both experimental and theoretical work. Static and dynamic experiments are conducted in a Plateau tank which makes a range of static Bond numbers accessible. The theoretical investigation includes both analytical and numerical approaches.

1.3 Background and significance

A liquid bridge, or captive drop, is a mass of liquid held by surface tension between two or more solid supports. Liquid bridges occur in a variety of physical and technological situations and a great deal of theoretical and experimental work has been done to determine axisymmetric equilibria for various disk configurations, bridge aspect ratios, axial gravity and rotations [1-18]. There have also been numerous investigations of the dynamics of axisymmetric liquid bridges subject to different excitations (impulses, vibration, etc.) [19-21]. Such investigations have been motivated both by practical considerations and basic scientific interest. Liquid bridges are important features in the positioning of liquid masses using surface tension forces. In crystal growth, they are associated with the floating-zone growth technique. Their oscillation and decay properties can also be used for viscosity and surface tension measurements of molten materials at high temperatures [22]. Pendular liquid bridges occur widely in the powder technology industry and are a major influence on powder flows and their mechanical properties [23]. In porous media flow, liquid-liquid displacement can lead to evolution of pendant and sessile lobes or lenticular bridges. The formation of liquid bridges from the gel that coats lung micro-airways results in occlusion of the bronchioles and is a precursor to respiratory problems and lung collapse [24].

Most theoretical and experimental work deals with liquid bridges that are anchored to the sharp edges of circular support disks. Slobozhanin and Tyuptsov [25] showed that for a bridge anchored to a sharp edge, the contact angle ϕ can vary freely within the range $\phi_e \leq \phi \leq \phi_e + \pi - \delta$

where ϕ_e is the wetting angle of the liquid in contact with the smooth solid surface, δ is the acute angle defined by the sharp edge. This pinning or anchoring of the contact line is sometimes referred to as canthotaxis [26] and allows the bridge to take on a wider range of stable configurations than would be admissible if the contact angle was restricted to be the equilibrium angle (as it would be for a flat or some other smooth surface). Bridges held between other types of supports have also been considered, including two flat plates [7, 27-30] a plate and a sphere [31] and two spheres [23, 32-34].

The shapes and stability of liquid bridges are governed by the following dimensionless numbers

$$\begin{aligned}
 Bo &= \Delta\rho R^2 g / \gamma && \equiv \text{Bond number} \\
 V &= V_0 / \pi R^2 L && \equiv \text{relative volume} \\
 \Lambda &= L / 2R && \equiv \text{slenderness} \\
 We &= \Delta\rho R^3 \Omega^2 / \gamma && \equiv \text{Weber number} \\
 K &= R_1 / R_2 && \equiv \text{Ratio of supporting disk radii.} \\
 \phi_1, \phi_2 &&& \equiv \text{lower and upper contact angles}
 \end{aligned}$$

Here $\Delta\rho$ is the density difference between the liquid bridge and the surrounding liquid or gas, R is the characteristic length associated with the bridge (usually the radius of the supporting disk), g is the gravitational acceleration, γ is the surface (or interfacial) tension, L is the distance between the disks, V_0 is the actual liquid volume and Ω is the angular rotation rate of the disks. Bo is a measure of the ratio of buoyancy to surface tension forces. The Weber number represents a balance between centrifugal and surface tension forces.

Our research has contributed to the fundamental understanding of fluid surface equilibria dynamics as well as providing a foundation for advances in related research areas (for example in crystal growth related problems). Liquid bridges have been involved in a number of past microgravity and our research results will be useful for the quantitative assessment of g-jitter effects on such experiments.

2. Accomplishments

2.1 Summary

The research involved an experimental and theoretical investigation of the statics and dynamics of nonaxisymmetric bridge configurations and nonaxisymmetric oscillations of initially axisymmetric bridges. The effects of nonaxial acceleration on the minimum volume stability limit have been investigated numerically. The minimum volume stability limit was found to be substantially modified as the lateral component of acceleration is increased. It appeared to approach, but not meet, the "zero Bond number" minimum volume limit for purely axial acceleration. This analysis was extended to include an investigation of maximum volume limits and was the subject of two journal publications (papers 3 and 4 listed in section 3.1)

We also investigated equilibrium stability problems for multiparametric axisymmetric equilibrium states (papers 2 and 7, section 3.1). The stability of a bridge subject to the combined effect of an axial gravity and isorotation was considered for restrictions typical for the floating zone technique. The influence of unequal disk radii combined with axial gravity on the entire boundary of stability with respect to arbitrary perturbations has also been analyzed. The work on isorotating bridges has been completed and submitted for publication (paper 7, section 3.1). Experimental and theoretical work continues on the unequal disk radii problem under the newly funded grant

NAG3-1684 that started in May of 1996. We also undertook theoretical and experimental investigations on the behavior of weightless bridges when its axisymmetric shape loses stability (paper 6, section 3.1).

For coaxial disk supports, the stability of nonaxisymmetric bridges subject to axial gravity has also been investigated experimentally and through numerical simulation (preliminary results are discussed in paper 3 section 3.1). This is the first attempt to determine the stability limit of nonaxisymmetric bridges for ranges of relative volumes and aspect ratios beyond the maximum volume limit for axisymmetric bridges that we are aware of.

2.2 Experimental

2.2.1 Experimental Set-up

Figure 1 schematically depicts our Plateau chamber. Liquid bridges are formed between rigid sharp-edged 1 cm diameter circular disks. The disks are mounted on supports that allow for independent rotation and lateral and vertical translation. These motions are facilitated through two 3-axis precision motor/drive systems. This provides for vertical oscillation, rotation and small amplitude lateral oscillation (the slip-ring gasket constrains the allowable lateral motion of the lower disk). The upper disk is supported by an injection tube. The disks are made from stainless steel. The bridge liquid is injected or removed through an injection tube which terminates in a 4mm-diameter hole in the center of the upper disk. A calibrated microsyringe is used for the injection of a fixed volume of liquid. The bridge is simultaneously lengthened by slowly moving the disks apart to the required separation distance. This distance can be determined to within 1-2 μm . A 3-way purge valve is suitably positioned to trap air bubbles.

Each support can be independently vibrated at frequencies less than 10 Hz. We use two imaging methods. Video images are obtained from two orthogonal cameras. A high quality Fourier transform imaging system is used for edge detection.

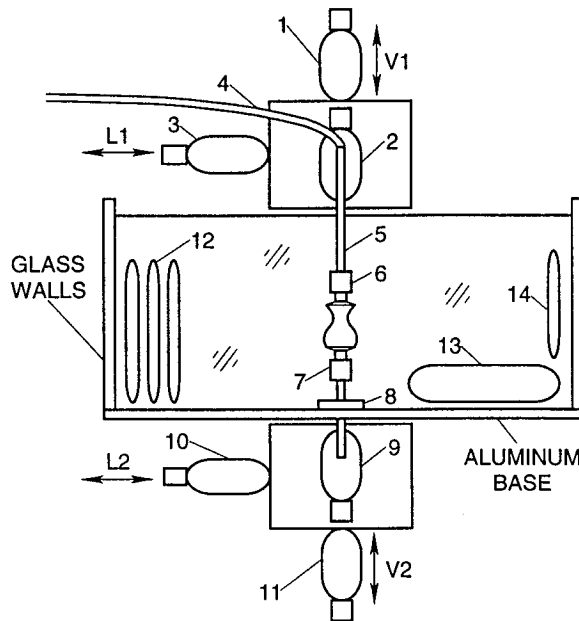


Fig. 1 Plateau Chamber: (1) upper vertical displacement motor V1; (2) upper rotational motor; (3) upper lateral displacement motor, L1; (4) bridge fluid injection line; (5) upper spindle; (6) upper feed disk; (7) lower feed disk; (8) slip-ring gasket; (9) lower rotational motor; (10) lower lateral displacement motor, L2; (11) lower vertical displacement motor V2; (12) cooling coils; (13) bath circulator (14) heating coil.

The disk diameters are known to within 10 μm . The length of the bridge is set by the positioning device and can be determined with a precision of 1-2 μm . Thus, for bridges of 2.5 cm

length the slenderness ($L/2R_0$), can be determined to within $\pm 0.04\%$. Volume can be measured with a precision of 0.5 mm^3 and an accuracy of 0.1% .

The liquid bath is a methanol-water solution. Variation of the methanol concentration changes the density difference between the Dow Corning 200[®] silicone oil bridge and the bath. We control the both the bath temperature and the methanol concentration to adjust Bo . At 22.5°C and 83% water concentration a condition of neutral buoyancy is obtained. Temperature control is crucial [26] since typical relative density changes with temperature are $2 \times 10^{-5} \text{ g cm}^{-3} \text{K}^{-1}$. Thus, to achieve density matches to within $10^{-4} \text{ g cm}^{-3}$ requires temperature control to within $\pm 0.1 \text{ K}$. This is achieved by control of the ambient temperature [26]. Two orthogonally positioned CCD cameras are used for visualization. They are connected to a framegrabber and VCR for data recording. One viewing axis uses incoherent white light and is used to obtain an overall image of the bridge. The second axis uses coherent light from a He-Ne laser. The image of the bridge is optically processed by placing a high pass filter at the Fourier transform plane. Thus, only the edges of the bridge pass through the filter. A series of lenses are then used to magnify the bridge. The lenses provide an angular resolution of 0.5 arcseconds which yields a resolution of $5 \mu\text{m}$ in the object plane. This allows accurate viewing of the bridge shape down to scales where “microscopic” contact angle behavior can be observed.

2.2.2 Experiments

Experiments have been carried out in three areas.

a) *Lateral shearing, squeezing and force measurements.*

For a liquid bridge held between two flat coaxial equidimensional circular disks, we used a force deflection apparatus to measure the total force exerted by a liquid bridge on the lower disk. The lower disk is attached to the end of a long cantilever arm. The cantilever arm acts as a weak spring. The liquid bridge displaces the disk by an amount proportional to the total force exerted on the disk. Then, by directly measuring the displacement, we can determine the total force. After calibrating the deflection of the cantilever arm with known applied loads, two sets of experiments were performed. In the first set, the total force was measured for various aspect ratio right circular cylindrical bridges. It is easy to calculate the forces exerted by right circular cylindrical bridges, and to verify the measurement by comparison with theoretical predictions. The second set of experiments involves measuring the force exerted by liquid bridges with a variety of aspect ratios and volumes at different Bond numbers. The measurements have been compared to theoretical predictions for various relative volumes and aspect ratios and agreement is good. It was shown that a minimum in capillary pressure does not always result in a maximum upward force on the lower disk. In addition, depending on the actual bridge volume, when the aspect ratio is increased the bridge may break before a minimum in capillary pressure is experienced. This work has not yet been submitted for publication but has been described in detail in previous annual reports.

b) *Stability limits for symmetric and nonsymmetric bridges*

This work has involved the determination of the stability limits for large volume ($V \gg 1$) nonaxisymmetric bridges subject to axial acceleration and was motivated by our theoretical investigation (see the end of section 2.3.2). Experimental work involved the characterization of stability limits of nonaxisymmetric bridges in an axial gravity field as determined numerically. The maximum volume stability limit was also investigated for axisymmetric bridges. A stability analysis (see paper 6 section 3.1) showed the existence of a bifurcation point at a critical slenderness, Λ_c , along the upper stability margin. Below Λ_c , the transition from an axisymmetric to a nonaxisymmetric configuration is continuous. However, above Λ_c , the transition is

discontinuous. This was investigated experimentally and the results are described in a publication (paper 6, section 3.1).

(c) *Vibration dynamics*

We have undertaken an experiment to determine the spatial mode structure of initially right circular cylindrical liquid bridges subject to lateral sinusoidal oscillation. We have developed a coherent imaging optical system that uses a Fourier transform of the image of a liquid bridge silhouette. By blocking the zero frequency, or dc component of the optical scene, only the high-frequency components corresponding to edges are passed through the system. This approach was used for several reasons. The system is designed for the study of contact line and angle dynamics and their relation to the larger scale dynamics of the liquid bridge surface. The advantages of this approach are that: (1) Optical filtration allows us to more precisely measure this angle by accurately locating the edge of the bridge. (2) By performing some of the image processing optically, we eliminate the need for expensive and specialized image processing hardware. (3) For liquid bridges undergoing sinusoidal deformations, the transform plane is a convenient location to obtain vibrational mode information directly.

Previous studies have focused on determining the resonant frequencies. We examined the nonlinear behavior of the forced oscillation of liquid bridges that act as nonlinear weak springs. At the limit of small vibration amplitudes, the response of the liquid bridge to vibration is typically Lorentzian, centered about the resonance frequency. However, at higher vibration amplitudes, nonlinear characteristics of the liquid bridge appear to become important. This work is still in progress.

2.3 Theoretical

2.3.1 Dynamics: Oscillation, vibration and breaking

A joint study carried out at the CMMR and our colleagues at the Laboratorio Aerodinámica E.T.S.I. Aeronáuticos (*LAMF*) at the Universidad Politécnica de Madrid focused on the effect of vibration on the stability limits of bridges and the modification of the static stability boundaries (see paper 1 section 3.1). The analysis was based on a self-similar one-dimensional model in which the axial velocity is assumed to be dependent on the axial coordinate z and the time t , but not on the radial coordinate r . Within the validity range of this analysis, the time variation of the interface deformation satisfies Duffing's equation. The results indicated that, depending on the type of axial vibration, stabilization or destabilization relative to the static stability margin can occur.

We have also developed a numerical method for the description of three dimensional oscillations of liquid bridges. It is a modified volume of fluid/continuum surface force (VOF/CSF) method. The method is suitable for liquid fragmentation and coalescence problems for arbitrary shaped fluid surfaces and is well suited for situations that involve a second surrounding fluid (liquid or vapor). We have applied it to axisymmetric and nonaxisymmetric breaking problems including the problem of nonaxisymmetric vibration. Preliminary results are discussed in a publication (paper 4 of section 3.1) and a manuscript describing the method is in preparation.

2.3.2 Stability of nonaxisymmetric liquid bridges

In this work we examined large nonaxisymmetric deformations of liquid bridges (with volume V_0 , held between equidimensional coaxial disks of radius R and separated by a distance L). First, we consider a numerical analysis (using *Surface Evolver*) of the stability limits of liquid bridges subject to lateral accelerations as a function of slenderness ($\Lambda = L/2R$), relative volume ($V = V_0/\pi R^2 L$), and Bond number ($Bo = \Delta \rho g R^2 / \gamma$). For axisymmetric bridges subject to axial

gravity, these stability limits correspond to a situation when the axisymmetric bridge breaks, or when the axisymmetric bridge loses stability to a stable nonaxisymmetric shape. The lateral acceleration stability limit is defined in terms of loss of stability by breaking. This limit is determined for both large and small values of the relative volume. The stability limit can be divided into two basic segments. One segment appears to be indistinguishable from part of the margin for the zero-Bond number case. The other segment belongs to a one-parameter (Bo) family of curves which, for a given Bo and a fixed value of Λ , have a maximum and minimum stable relative volume. Each of these curves is asymptotic to another part of the minimum volume zero-Bond number limit up to a point determined by the particular value of Bo and has a turning point corresponding to a maximum value of Λ . For $V \gg 1$, the maximum stable volume tends to infinity as $\Lambda \rightarrow 0$. For any given lateral Bond number, the minimum volume stability limit is decreased and becomes indistinguishable from the zero Bond number limit when Λ becomes sufficiently small. For unstable bridges in the vicinity of the stability limit a consistent sequence of shapes can be readily identified and are recognizable by their overall shape and the number of necks they possess.

In a recent bifurcation analysis for $V = 1$ bridges subject to lateral gravity (Laveron et al. 1995) it was speculated that, because the eigenfunction associated with a subcritical bifurcations for $\Lambda > \Lambda_c$ is antisymmetric with respect to the $z = 0$ plane, the bridge would break into two drops of unequal volumes. Likewise it was speculated that, for $\Lambda < \Lambda_c$, loss of stability would lead to equal size drops since destabilization occurs through a turning point and the associated eigenfunction is symmetric. Our results confirm this, although we note that, when breaking occurs, three drops form, two which remain attached to the disks and a third, free satellite drop. Our results for other values of V are discussed in terms of the $V=1$ case.

The stability of axisymmetric liquid bridges subject to axial gravity has been discussed in [17] and it is well-known that axisymmetric bridges are unstable to nonaxisymmetric perturbations which can lead to stable nonaxisymmetric bridges. However, little is known about the stability of these nonaxisymmetric configurations. We examined the stability of these bridges is numerically (using *Surface Evolver*) for $Bo = 0.1$ and 2, and experimentally. We find that the maximum volume segment of the stability limit appears to follow the same trend as for lateral gravity. We conjecture that, as for lateral gravity, the maximum stable relative volume, V tends to infinity as the slenderness, Λ , tends to zero. Our results are described in two publications (papers 3 and 4 of section 3.1).

2.3.3 Bifurcation of the equilibrium states of a weightless liquid bridge

The bifurcation of the solutions of the nonlinear equilibrium problem of a weightless liquid bridge with a free surface pinned to the edges of two coaxial equidimensional circular disks was examined. The bifurcation is studied in the neighborhood of the stability boundary for axisymmetric equilibrium states. The first approximations for the shapes of the bifurcated equilibrium free surfaces were obtained. These surfaces exhibit a variety of shapes that depend on the nature of the perturbation that leads to loss of stability of the axisymmetric states. From the structure of the bifurcation the stability of the bifurcated equilibrium shapes was determined. Loss of stability with respect to axisymmetric perturbations always leads to breaking of the bridge. Along the maximum volume stability limit, depending on values of the system parameters, loss of stability with respect to nonaxisymmetric perturbations results in either a jump or a continuous transition to stable nonaxisymmetric shapes. The value of the slenderness at which a change in the type of transition occurs is found to be $\Lambda_s = 0.4946$. Experimental

investigation based on a neutral buoyancy technique agrees with this prediction. It shows that for $\Lambda < \Lambda_S$, the jump is finite and that a critical bridge undergoes a finite deformation to a stable nonaxisymmetric state. Results are described in a publication (paper 2 section 3.1.)

2.3.4 Stability of an isorotating liquid bridge in an axial gravity field

The stability problem for the axisymmetric equilibrium states of an isorotating liquid bridge between equidimensional circular disks in a constant axial gravity field was considered. In particular, we examined the stability of bridges satisfying two types of constraint that are typical for the floating zone method used for materials purification and single crystal growth. First we consider the constraint that the relative volume of the bridge, V , is equal to 1. For this case, the critical values of the slenderness (Λ) and of the liquid contact angles at both disks have been determined for a wide range of the Bond (B) and Weber (W) numbers. The second constraint is that the liquid contact angle at one of the disks is prescribed (the chosen values are 0° and 15° and correspond to extremes in the range of growing angle values). For this case, the dependencies of critical Λ and V values on B and W have been calculated. In addition, both axial gravity directions are considered separately and for critical states, the values of the contact angle at the other disk are also analyzed. (Results are described in paper 2 of section 3.1.)

3. Publications and presentations

3.1 Publications

1. J. Meseguer, M.A. González and J.I.D. Alexander, "Dynamic stability of long axisymmetric liquid bridges", *Microgravity Science and Technology*, 7, 234-242. 1994.
2. L. Slobozhanin, J.I.D. Alexander and A. Resnick, "Stability of liquid bridges under reduced gravity", to appear, Proceedings of the 2nd Symposium on Fluids in Space, Naples, Italy, April 22-26, 1996.
3. J.I.D. Alexander, "Stability of nonaxisymmetric liquid bridges" (to appear) *Microgravity Science and Technology*, (1997)
4. J.I.D. Alexander, Y. Zhang, S. Delafontaine and A. Fedoseyev, "Numerical Simulation of Liquid Bridge Dynamics and Statics", to appear in *Izv. Akad. Nauk Mekh. Zhid. Gaza* (Fluid Dynamics), 1997.
5. A. Resnick and J.I.D. Alexander, "A Plateau Tank Apparatus for the Study of Liquid Bridges", to appear *Review Scientific Instruments* 68 (1997).
6. L. Slobozhanin, J.I.D. Alexander and A. Resnick, "Bifurcation of the equilibrium states of a weightless liquid bridge", (in review) *Physics of Fluids*, 1996.
7. L. Slobozhanin, J.I.D. Alexander, "Stability of an isorotating liquid bridge in an axial gravity field", (in review) *Physics of Fluids*, 1996.
8. D. Lyubimov, T. Lyubimov, J.I.D. Alexander and N.I. Lobov, "On the Boussinesq approximation for fluid systems with deformable boundaries", submitted to *Journal of Fluid Mechanics*, 1996.
9. A. Resnick and J.I.D. Alexander, "A Coherent Fourier Imaging System for Liquid Bridge Experiments" (manuscript in preparation).

3.2 Presentations

- A. Resnick, L. Slobozhanin and J.I.D. Alexander, "Influence of support geometry and gravity on the stability of liquid bridges Division of Fluid Mechanics, American Physical Society, Syracuse, New York, November 1996.
- J.I.D. Alexander, "Large deformations of liquid bridges", Invited presentation at The Second Symposium on Fluids in Space, Naples, Italy, April 22-26, 1996.
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- J.I.D. Alexander, "Scaling and estimation of experiment response to g-jitter," presented at ESA Physical science working group meeting, ESA HQ, Paris France, September 1994.
- J.I.D. Alexander, "Stability of nonaxisymmetric liquid bridges", European Low Gravity Research Association Biannual Meeting (ELGRA), Madrid, Spain, December 11-14, 1994.

4. Students supported under this grant

Two students were supported under this grant: Andrew Resnick (Ph.D, Physics, UAH). and Sylvie Delafontaine (Visiting MS student, Mediterranean Institute of Technology, Marseille, France). Mr. Resnick is expected to graduate in June, 1997.

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Appendix: Abstracts of the papers listed in Section 3.

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- J.I.D. Alexander, "Large deformations of liquid bridges", Invited presentation at The Second Symposium on Fluids in Space, Naples, Italy, April 22-26, 1996.
- L. Slobozhanin, J.I.D. Alexander and A. Resnick, "Stability of Liquid bridges under reduced gravity", to appear, Proceedings of the 2nd Symposium on Fluids in Space, Naples, Italy, April 22-26, 1996
- J.I.D. Alexander, "Scaling and estimation of experiment response to g-jitter," presented at ESA Physical science working group meeting, ESA HQ, Paris France, September 1994.
- J.I.D. Alexander, "Stability of nonaxisymmetric liquid bridges", European Low Gravity Research Association Biannual Meeting (ELGRA), Madrid, Spain, December 11-14, 1994.

4. Students supported under this grant

Two students were supported under this grant: Andrew Resnick (Ph.D, Physics, UAH). and Sylvie Delafontaine (Visiting MS student, Mediterranean Institute of Technology, Marseille, France). Mr. Resnick is expected to graduate in June, 1997.

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J. Meseguer, M. A. González, and J. I. D. Alexander

Dynamic Stability of Long, Axisymmetric Liquid Bridges

This paper deals with the non-linear forced oscillations of axisymmetric long liquid bridges between equal disks. The dynamics of the liquid bridge has been analyzed by using a self-similar, one-dimensional model already used in similar problems. The influence of the dynamics on the static stability limits, as well as the main characteristics of the non-linear behaviour of long liquid bridges, have been studied within the range of validity of the mathematical model used here.

1 Introduction

In the last years isothermal liquid bridges have focused the attention of many investigators, with a large number of published papers on this topic (a review of the literature related to liquid bridges can be found in Sanz-Andres [1]). Because of the large number of parameters involved, the study of liquid bridges becomes a formidable task which extends in time from the early paper of Gillette and Dyson [2] to the late paper of Slobozhanin and Perales [3]. Most of these papers are only concerned with static stability limits. As far as we know only a few attempts have been made to analyze the influence of the dynamics of the liquid bridge, and these efforts have been centred more in the dynamics itself than in its influence on the stability limits (Meseguer [4], Rivas and Meseguer [5], Perales and Meseguer [6], Sanz and López-Díez [7], Zhang and Alexander [8], Langbein [9], Schulkes [10], among others).

In this paper the influence of the dynamics on the stability limits of liquid bridges is analyzed by using a simplified one-dimensional, self-similar model which, in spite of its simplicity, allows us to get some feeling on the dynamical behaviour of long liquid bridges. Associated with stability limits is the concept of stability margin, which has been defined as the difference between the energy of the stable equilibrium shape and of the unstable one for a given liquid bridge. The stability margin gives an estimate of the minimum energy which is needed to break a liquid bridge

through a given perturbation. Stability margins are discussed in sect. 2, whereas in sect. 3 the forced oscillations of long liquid bridges are analyzed, and stability diagrams for such kinds of perturbation are obtained.

2 Problem Formulation

The fluid configuration under consideration consists of an axisymmetric liquid bridge, with constant density ρ , kinematic viscosity ν and surface tension σ , held by capillary forces between two coaxial, solid disks of radius R which are a distance L apart, as sketched in fig. 1. The volume of liquid, V' , is assumed to be almost the volume of a cylinder of the same R and L , $V' \approx \pi R^2 L$, and it is also assumed that there is a small gravity field acting parallel to the liquid bridge axis. To describe the behaviour of such a fluid configuration the following dimensionless parameters are introduced: slenderness $\Lambda = L/(2R)$, dimensionless volume $V = V'/R^3$, Bond number $B = \rho g R^2/\sigma$, and viscosity parameter $C = \nu(\rho/(\sigma R))^{1/2}$. In addition to these parameters, it must be stated that all physical magnitudes used in the following have been made dimensionless by using R and $(\rho R^3/\sigma)^{1/2}$ as characteristic length and characteristic time, respectively.

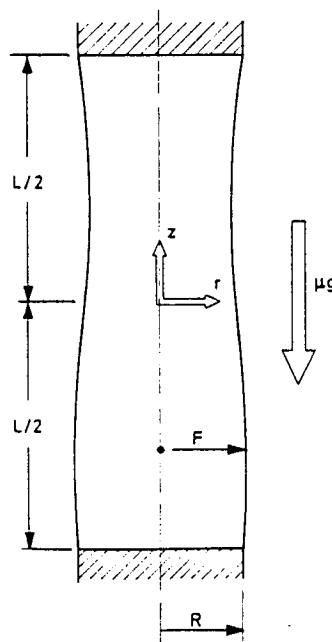


Fig. 1. Geometry and coordinate system for the liquid bridge problem

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Paper submitted: May 25, 1993.

Submission of final revised version: December 2, 1993.

Paper accepted: January 7, 1994.

STABILITY OF LIQUID BRIDGES UNDER REDUCED GRAVITY

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Key words: capillarity, equilibrium, stability, liquid bridge

Abstract

The results of two lines of investigation of the equilibrium stability problems for a capillary liquid bridge between two coaxial circular disks are presented. The first deals with the determination of the stability regions for multiparametric axisymmetric equilibrium states. The stability of a bridge subject to combine effect of an axial gravity and isorotation is considered when restrictions typical for the floating zone technique are accounted for. The influence of unequal disk radii combined with an axial gravity on the entire boundary of stability with respect to arbitrary perturbations is also analyzed. The second line concerns the theoretical and experimental investigations on the behavior of the weightless bridge when its axisymmetric shape loses stability.

Stability of nonaxisymmetric liquid bridges^{*}

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Abstract

The stability of nonaxisymmetric liquid bridges held between equidimensional coaxial disks of radius R and separated by a distance L is examined. The stability limits for lateral and axial acceleration are considered. The lateral acceleration stability limit is defined in terms of loss of stability by breaking. This limit is determined for both large and small values of the relative volume, V . The stability limit can be divided into two basic segments. One segment appears to be indistinguishable from part of the margin for the zero-Bond number case. The other segment belongs to a one-parameter family of curves which, for a given Bond number and a fixed value of slenderness $\Lambda = L/2R$, have a maximum and minimum stable relative volume. The maximum volume stability limit tends to infinity as $\Lambda \rightarrow 0$. For any given lateral Bond number, the minimum volume stability limit is decreased and becomes indistinguishable from the zero Bond number limit when Λ becomes sufficiently small.

^{*} This paper is dedicated to Professor Dieter Langbein on the occasion of his retirement in recognition of his fundamental contributions to the understanding of liquid bridges and other phenomena associated with liquid surfaces.

Numerical Simulation of Liquid Bridge Dynamics and Statics

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Abstract

The choice of numerical method for the simulation of liquid bridge dynamics and statics will depend on the geometry of the bridge (slenderness, volume, support geometry, surface shape) as well as the particular nature of the dynamical or static states under consideration. We examine a number of techniques ranging from finite-difference methods which treat surface boundary conditions directly, to Volume-of Fluid methods and an energy minimization method suitable for computation of static equilibrium liquid bridge shapes for liquid bridges held between coaxial equidimensional disks. Methods which do not involve mapping transformations only work well when the surface of the bridge does not become a multivalued function of the coordinate system in which the bridge is described. This generally restricts their application to bridges with volumes of less than twice the volume the right circular cylinder that would fit between the two support disks. In general, we found that the *Surface Evolver* method which involves a global energy minimization worked well over a large range of volumes and that breaking and oscillation dynamics of large volume bridges was simulated well using a modified Volume-of-Fluid method.

A Plateau tank apparatus for the study of liquid bridges

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Abstract:

An apparatus has been constructed and used to study the equilibrium and dynamical behavior of liquid bridges under reduced effective gravity. Liquid bridges are created and manipulated using 6 independent computer controlled stepper motors which drive linear motion tables. The bridges are visualized with a high magnification coherent Fourier optical system and in an orthogonal view using incoherent white light. By calibrating a density hydrometer and measuring the interfacial energy between the bridge and bath, reliable Bond numbers as low as 10^{-4} can be created and held stable for extended periods of time. Dimensional control of the liquid bridges approaches one part in 10^{-4} for the aspect ratio and volume. The apparatus has been tested by measuring the static stability limits of axisymmetric bridges and comparing the results with previous theoretical predictions. Experimental error for the apparatus is $\delta Bo/Bo = 0.02$, $\delta \Lambda/\Lambda = 0.001$ and $\delta V/V = 0.001$, where Bo is the Bond number, Λ is the aspect ratio of the bridge and V is the dimensionless (relative) volume of the bridge.

BIFURCATION OF THE EQUILIBRIUM STATES OF A WEIGHTLESS LIQUID BRIDGE[†]

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The bifurcation of the solutions of the nonlinear equilibrium problem of a weightless liquid bridge with a free surface pinned to the edges of two coaxial equidimensional circular disks is examined. The bifurcation is studied in the neighborhood of the stability boundary for axisymmetric equilibrium states with emphasis on the boundary segment corresponding to nonaxisymmetric critical perturbations. The first approximations for the shapes of the bifurcated equilibrium surfaces are obtained. The stability of the bifurcated states is then determined from the bifurcation structure. Along the maximum volume stability limit, depending on values of the system parameters, loss of stability with respect to nonaxisymmetric perturbations results in either a jump or a continuous transition to stable nonaxisymmetric shapes. The value of the slenderness at which a change in the type of transition occurs is found to be $\Lambda_s = 0.4946$. Experimental investigation based on a neutral buoyancy technique agrees with this prediction. It shows that, for $\Lambda < \Lambda_s$, the jump is finite and that a critical bridge undergoes a finite deformation to a stable nonaxisymmetric state.

[†] Dedicated to the memory of Dr. A.D. Tyuptsov.

STABILITY OF AN ISOROTATING LIQUID BRIDGE IN AN AXIAL GRAVITY FIELD

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The stability problem for the axisymmetric equilibrium states of an isorotating liquid bridge between equidimensional circular disks in a constant axial gravity field is considered. In particular, we examine the stability of bridges satisfying two types of constraint that are typical for the floating zone method used for materials purification and single crystal growth. First we consider the constraint that the relative volume of the bridge, V , is equal to 1. Here, V is the ratio of the actual bridge volume to that of a cylinder pinned to the edges of disks. For this case, the critical values of the slenderness (Λ) (ratio of the disk separation to the diameter) and of the liquid contact angles (β_1, β_2) at both disks have been determined for a wide range of the Bond (B) and Weber (W) numbers. The second constraint is that the liquid contact angle β_1 at one of the disks is prescribed (the chosen values are 0° and 15° and correspond to extremes in the range of growing angle values). For this case, the dependencies of critical Λ and V values on B and W have been calculated. In addition, both axial gravity directions are considered separately and the values of the contact angle at the other disk, β_2 , for critical states are also analyzed.

The solution of the stability problem for any liquid bridge is discussed in detail using the case for $B = W = 0.1$ as an example. In particular, the relationship between the general boundary of the stability region and the stability of bridges subject to the constraints outlined above is examined.

ON THE BOUSSINESQ APPROXIMATION FOR FLUID SYSTEMS WITH DEFORMABLE INTERFACES

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ABSTRACT

In this paper we reconsider the limits in which the Boussinesq approximation is valid for fluid systems with deformable interfaces and develop a generalization of the approximation which allows for self-consistent application to such systems. The Boussinesq limit is characterized by two parameters, G and ϵ , where G is a dimensionless measure of gravitational acceleration or system size and ϵ is the product of the fluid's coefficient of thermal expansion with a characteristic temperature difference. In general, the Boussinesq limit corresponds to $G \rightarrow \infty$ and $\epsilon \rightarrow 0$ while the product $G\epsilon$ (equal to the familiar Grashof or Rayleigh numbers, depending on the chosen scale factors) remains finite. We consider three problems involving deformable boundaries: the stability of a two-layer fluid system heated from above and below; the equilibrium of an encapsulated liquid drop in vertical temperature gradient; and the influence of buoyancy on long-wavelength Marangoni instability. In the first two problems, we examine the conditions required to consistently account for the effects of a deformable surface on thermal convection while simultaneously applying the Boussinesq approximation. In particular, the effects of the deformable surface can be included through a term proportional to $G\delta\zeta$, where δ is the dimensionless density difference across the surface and ζ is the deflection of the interface from planarity. However, for consistency, it is then required the density of the two fluids be treated as equal in the equations of motion